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On a recursive equation over a p-adic field

Farrukh Mukhamedov [1]

Departamento de Fisica, Universidade de Aveiro, Campus Universitário de Santiago, 3810-193 Aveiro, Portugal

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Abstract

In this work we completely describe the set of all solutions of a recursive equation arising from the Bethe lattice models over p-adic numbers.

Keywords: Recursive equation; p-Adic numbers; Model; Fixed point; Unique

1. Introduction

The p-adic numbers were introduced by K. Hensel. For about a century after the inventing of the p-adic numbers, they were mainly considered objects of pure mathematics. After discovering that the physics of certain models could be based on the idea that the structure of space-time for very short distances might conveniently be described in terms of p-adic numbers, many applications of such numbers in theoretical physics have been proposed in various works (see for example, [4,6,12,15]). A number of p-adic models in physics cannot be described using ordinary probability theory based on the Kolmogorov axioms; therefore p-adic probability models were investigated in [7]. Such models appear to provide the probabilistic interpretation of p-adic valued wavefunctions and string amplitudes in the framework of p-adic theoretical physics (see [6,7]). Using such a p-adic probability approach in [13,14] we have developed a theory of statistical mechanics in the context of that theory of probability. In statistical mechanics, due to their solvable character [1], Bethe lattice models and models on random tree-like graphs [3] play a central role. In the calculation of the average magnetization of these models there naturally appears the following recursive equation:

$$h_n = \log\left[\left(\frac{e^{\alpha_{n+1}+h_{n+1}} + e^{\beta_{n+1}}}{e^{\gamma_{n+1}} + e^{h_{n+1}}}\right)\left(\frac{e^{\alpha_{n+2}+h_{n+2}} + e^{\beta_{n+2}}}{e^{\gamma_{n+2}} + e^{h_{n+2}}}\right)\right], \quad n \in \mathbb{N}, \tag{1.1}$$

where $\alpha_k, \beta_k, \gamma_k \in \mathbb{R}$ and $\{h_n\}$ is an unknown sequence of real numbers; the solution of such an equation describes corresponding Gibbs measures [10]. On certain conditions depending on parameters $\alpha_k, \beta_k, \gamma_k$, Eq. (1.1) has infinitely many solutions [2,5] (i.e. over real numbers), but the whole set of solutions is still not described.

E-mail addresses: far75m@yandex.ru, farruh@fis.ua.pt.
[1] On leave from: Department of Mechanics and Mathematics, National University of Uzbekistan, Tashkent, 700174, Uzbekistan.

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In the present work we are going to completely describe the set of solutions of (1.1) over p-adic numbers. Our hope is that this allows us further understanding of the Bethe lattice models and models on random tree-like graphs over p-adic numbers. Moreover, we think that the result obtained will give certain information on random rational p-adic dynamical systems generated by linear-fractional functions. Note that some applications of p-adic dynamical systems to some biological and physical systems have been proposed in [8].

Throughout the work p will be a fixed prime number greater than 3, i.e. $p \geq 3$. Every rational number $x \neq 0$ can be represented in the form $x = p^r \frac{n}{m}$, where $r, n \in \mathbb{Z}$, m is a positive integer, $(p, n) = 1$, $(p, m) = 1$. The p-adic norm of x is given by

$$|x|_p = \begin{cases} p^{-r} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0. \end{cases}$$

It satisfies the following strong triangle inequality:

$$|x + y|_p \leq \max\{|x|_p, |y|_p\},$$

which is a non-Archimedean norm.

The completion of the field of rational numbers $\mathbb{Q}$ with respect to the p-adic norm is called the *p-adic field* and it is denoted by $\mathbb{Q}_p$.

Given $a \in \mathbb{Q}_p$ and $r > 0$ put

$$B(a, r) = \{x \in \mathbb{Q}_p : |x - a|_p < r\}, \qquad S(a, r) = \{x \in \mathbb{Q}_p : |x - a|_p = r\}.$$

The *p-adic logarithm* is defined by the series

$$\log_p(x) = \log_p(1 + (x - 1)) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x - 1)^n}{n},$$

which converges for $x \in B(1, 1)$; the *p-adic exponential* is defined by

$$\exp_p(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

which converges for $x \in B(0, p^{-1/(p-1)})$.

Lemma 1.1 *([9]). Let $x \in B(0, p^{-1/(p-1)})$; then we have*

$$|\exp_p(x)|_p = 1, \qquad |\exp_p(x) - 1|_p = |x|_p, \qquad |\log_p(1 + x)|_p = |x|_p \tag{1.2}$$

$$\log_p(\exp_p(x)) = x, \qquad \exp_p(\log_p(1 + x)) = 1 + x. \tag{1.3}$$

2. Main result

In this section we are going to completely describe the set of solutions of (1.1) over p-adic numbers. Let us first formulate the problem in a p-adic setting. Namely, we are interested in finding all solutions of the following recursive equation:

$$h_n = \log_p \left[\left(\frac{\exp_p(\alpha_{n+1} + h_{n+1}) + \exp_p(\beta_{n+1})}{\exp_p(\gamma_{n+1}) + \exp_p(h_{n+1})} \right) \left(\frac{\exp_p(\alpha_{n+2} + h_{n+2}) + \exp_p(\beta_{n+2})}{\exp_p(\gamma_{n+2}) + \exp_p(h_{n+2})} \right) \right], \quad n \in \mathbb{N}, \tag{2.1}$$

where $\alpha_k, \beta_k, \gamma_k \in \mathbb{Q}_p$, such that

$$|\alpha_k|_p < p^{-1/(p-1)}, \qquad |\beta_k|_p < p^{-1/(p-1)}, \qquad |\gamma_k|_p < p^{-1/(p-1)}, \tag{2.2}$$

and $h_k \in \mathbb{Q}_p$ also should satisfy $|h_k|_p < p^{-1/(p-1)}$ for every $k \in \mathbb{N}$, since these conditions ensure the existence of the p-adic $\log_p$ and $\exp_p$.

Define

$$S = \{\mathbf{h} = (h_n)_{n\in\mathbb{N}} : \ h_n \text{ satisfies } (2.1)\}.$$

Let us rewrite (2.1) as follows:

$$u_n = \left(\frac{a_{n+1}u_{n+1} + b_{n+1}}{c_{n+1} + u_{n+1}}\right)\left(\frac{a_{n+2}u_{n+2} + b_{n+2}}{c_{n+2} + u_{n+2}}\right), \quad n \in \mathbb{N}, \tag{2.3}$$

where we have defined

$$\left.\begin{array}{ll} a_k = \exp_p(\alpha_k), & b_k = \exp_p(\beta_k), \\ c_k = \exp_p(\gamma_k), & u_k = \exp_p(h_k) \end{array}\right\}, \quad k \in \mathbb{N}. \tag{2.4}$$

Let us introduce

$$f_k(x) = \frac{a_k x + b_k}{c_k + x}, \quad k \in \mathbb{N}. \tag{2.5}$$

The functions f_k have the following properties.

Lemma 2.1. *For every $k \in \mathbb{N}$ the following relations hold:*

$$|f_k(x) - f_k(y)|_p \le \frac{1}{p}\frac{|x-y|_p}{|c_k + x|_p|c_k + y|_p} \tag{2.6}$$

$$|f_k(\exp_p(h))|_p = 1 \quad \textit{for every } |h|_p \le \frac{1}{p}. \tag{2.7}$$

Proof. From (2.5) we immediately obtain

$$|f_k(x) - f_k(y)|_p = \frac{|x-y|_p|a_k c_k - b_k|_p}{|c_k + x|_p|c_k + y|_p}.$$

Keeping in mind (2.4) from (1.2), (2.2) one gets

$$|a_k c_k - b_k|_p = |\exp_p(\alpha_k + \gamma_k - \beta_k) - 1|_p \le \frac{1}{p}$$

which implies (2.6). The strong triangle inequality with (1.2) and $p \ge 3$ implies that $|\exp_p(h) + 1|_p = 1$; hence one finds (2.7). □

The main result of the work is the following

Theorem 2.2. *Let (2.2) be satisfied. Then $|S| \le 1$, where $|A|$ means the number of elements of a set A.*

Proof. If $S = \emptyset$, then there is nothing to prove. So, assume that $S \ne \emptyset$. To prove theorem it is enough to prove that any two elements of S coincide with each other. To show this it is enough to prove that for arbitrary $\varepsilon > 0$ and every $\mathbf{h} = (h_n, n \in \mathbb{N})$, $\mathbf{s} = (s_n, n \in \mathbb{N}) \in S$ and $n \in \mathbb{N}$ the inequality $|h_n - s_n|_p < \varepsilon$ is valid.

Let $\mathbf{h} = (h_n, n \in \mathbb{N})$, $\mathbf{s} = (s_n, n \in \mathbb{N}) \in S$ and $\epsilon > 0$ be an arbitrary number. Put $v_k = \exp_p(s_k)$. Let $n \in \mathbb{N}$ be an arbitrary number. Using (2.5), (2.3) and Lemma 2.1 we get

$$\begin{aligned} |u_n - v_n|_p &= |f_{n+1}(u_{n+1})f_{n+2}(u_{n+2}) - f_{n+1}(v_{n+1})f_{n+2}(v_{n+2})|_p \\ &\le \max\left\{|f_{n+1}(u_{n+1})|_p|f_{n+2}(u_{n+2}) - f_{n+2}(v_{n+2})|_p, |f_{n+2}(v_{n+2})|_p|f_{n+1}(u_{n+1}) - f_{n+1}(v_{n+1})|_p\right\} \\ &\le \frac{1}{p}\max\{|u_{n+1} - v_{n+1}|_p, |u_{n+2} - v_{n+2}|_p\}, \end{aligned} \tag{2.8}$$

where we have used the following equalities:

$$|c_k + u_k|_p = 1, \qquad |c_k + v_k|_p = 1, \quad \forall k \in \mathbb{N},$$

which immediately follow from (1.2).

Now take $n_0 \in \mathbb{N}$ such that $\frac{1}{p^{n_0}} < \varepsilon$. Then iterating (2.8), one gets

$$|u_n - v_n|_p \leq \frac{1}{p^{n_0}} < \epsilon.$$

The last inequality with (1.2) and (2.4) implies that

$$|h_x - s_x|_p = |u_n - v_n|_p < \epsilon.$$

This completes the proof. □

Remark. Note that the proved theorem provides some applications to the p-adic Bethe lattice models and models on random tree-like graphs. This would be a theme of our further investigations. We also think that the result can be applied for the study of dynamics of random rational p-adic dynamical systems. It should also be noted that a certain type of recursive equations in p-adic numbers were considered in [11]; they were related with transcendentality of the p-adic numbers. On the other hand, the method presented here used a functional approach which is different from that of [11], where a number theoretical method was presented.

Now consider several cases when S is non-empty.

Case 1. Let us assume that the following equality holds:

$$a_k + b_k = c_k + 1$$

for every $k \in \mathbb{N}$, which implies that Eq. (2.3) has a solution $u_n = 1$ for all $n \in \mathbb{N}$. According to Theorem 2.2 it is unique, i.e. $|S| = 1$.

Case 2. Now suppose that $a_k = a, b_k = b, c_k = c$ for all $k \in \mathbb{N}$. We are going to show the existence of a solution of (2.3). Let us search for a solution in the form $u_n = u$, $\forall n \in \mathbb{N}$, where u is unknown. Then (2.3) reduces to the equation $f(u) = u$, where

$$f(u) = \left(\frac{au + b}{c + u}\right)^2.$$

Put $D = S(0, 1) \cap B(1, p^{-1/(p-1)})$. Let us show that $f(D) \subset D$. Indeed, let $x \in D$; then $|x|_p = 1$, $|x - 1|_p < p^{-1/(p-1)}$. According to Lemma 1.1 this means that $x = \exp_p(y)$ for some $y \in B(0, p^{-1/(p-1)})$. Now by means of Lemma 2.1 we infer that $|f(x)|_p = 1$. The strong triangle inequality with

$$|a - 1|_p \leq \frac{1}{p}, \qquad |b - c| \leq \frac{1}{p}$$

implies that

$$|f(x) - 1|_p = \left|\frac{(a-1)x + b - c}{c + x}\right|_p \left|\frac{(a-1)x + b + c}{c + x}\right|_p \leq \frac{1}{p}. \tag{2.9}$$

Using the same argument following the proof of Lemma 2.1 one gets

$$|f(x) - f(y)|_p \leq \frac{1}{p}|x - y|_p \quad \text{for every } x, y \in D. \tag{2.10}$$

Thus from the inequalities (2.9) and (2.10) we obtain that f is a contraction of D; hence f has a unique fixed point $\zeta \in D$. Consequently, S is non-empty. Now Theorem 2.2 yields that $|S| = 1$.

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